BULLETIN

OF THE

Society for the Promotion of Engineering Education

Vol. I LANCASTER, PA., FEBRUARY, 1911 No. 7

EDITORIAL.

Looking Forward.—The annual convention has been scheduled for the latter part of June and at Pittsburgh. The days will, presumably, be the three at the beginning or end of the week of the 25th. The reports and papers already in sight give promise of an interesting meeting. The industries of the vicinity are of great variety so that they will appeal to the entire membership of the Society. The Program Committee will provide ample opportunity for the members to get together for social as well as technical profit.

A Permanent Fund.—The suggestion of the treasurer that a permanent fund be established and that the life membership fees from the nucleus of this, meets with the unanimous approval of the Council. There are, undoubtedly, many members who can afford to invest the sum of fifty dollars at one time and thus be rid of the bother of remitting annually for dues. With even the present nominal dues the life membership fees form the nucleus of this, meets with the unanimous If, therefore, any member contemplates taking out life membership it is important that he do so soon.

Second-class Postal Rates for the Bulletin.—Under the present plan of publication the BULLETIN is not eligible to

second-class rates. Either it must qualify as the organ of a scientific society without the privilege of carrying advertising, or it must rank as a commercial proposition, in which case it must have a "legitimate circulation." In the latter case any member of the Society must have the privileges of membership without being required to subscribe to the BULLETIN, and he must have these at a substantial reduction in fee from that paid by subscribers. The majority of the members of the Council favor the latter plan. There is some hope that Congress will so modify the postal law as to allow scientific societies to publish periodicals and at the same time to carry advertising. There are periodicals now enjoying this privilege which they secured before the law was as strictly enforced as now. If such favorable action is taken by Congress the BULLETIN will have no farther difficulty. In the meantime, however, it appears necessary for the Society to make some change in the constitution to enable us to secure the lower postal rate. The most feasible plan, which has the approval of a majority of the Council, consists in a reduction of the dues to $3.00 with a subscription price of $1.00 for the BULLETIN to those who pay in advance. For the information of the Council, it is requested that all members who will subscribe for the BULLETIN for next year, in case its publication is continued and in case there appears to be no other way to secure second-class rates, return to the secretary at once the provisional subscription card which will be sent out by him. This appears to be an increase in the dues of the Society, but it will be remembered that the present free subscription to the BULLETIN was not expected to be a permanent institution. The Council authorized the publication of the BULLETIN this year and instructed the Publication Committee to furnish it without cost until the practicability of publishing it had been determined by experiment. Obviously the Society cannot permanently furnish a periodical containing between 600 and 700 pages per year entirely free when it could barely pay running expenses without this. While advertising pays a part of the printing bill, it is doubtful it

this source of income will supply more than one-half the total cost of publication.

NEW MEMBERS.*

ELECTED DECEMBER 31, 1910.

BERGER, F. A., Instructor in Mechanical Engineering, Washington University, St. Louis, Mo..... 1910

BOLTON, F. C., Professor of Electrical Engineering, Agricultural and Mechanical College of Texas, College Station, Texas..... 1910

CALLEN, A. C., Instructor in Mining Engineering, Lehigh University, South Bethlehem, Pa..... 1910

LAMKE, G. W., Instructor in Electrical Engineering, Washington University, St. Louis, Mo..... 1910

MCGONIGLE, CHAS., Contracting Engineer, Millikin Bros., Inc. (of New York, N. Y.), 686 Irwing St., Portland, Ore..... 1910

MILLER, B. L., Professor of Geology, Lehigh University, South Bethlehem, Pa..... 1910

MORSS, EVERETT, 115 Commonwealth Ave., Boston, Mass..... 1910

NICHOLS, C. W., Instructor in Rhetoric, University of Minnesota, Minneapolis, Minn..... 1910

RICHARDSON, M. R., Instructor in Mathematics, North Carolina College of A. and M. Arts, West Raleigh, N. C..... 1910

RISLEY, W. J., Professor of Mathematics, James Millikin University, Decatur, Ill..... 1910

RUSSELL, L. K., Professor of Chemistry, Clarkson School of Technology, Potsdam, N. Y..... 1910

SWEETSER, E. O., Assistant Professor of Civil Engineering, Washington University, St. Louis, Mo..... 1910

APPLICANTS FOR MEMBERSHIP.

BERENTS, H., Engineer, Anhui Railways, Wuhu, China..... 1911

CRATHORNE, A. R., Associate in Mathematics, University of Illinois, Champaign, Ill..... 1911

CROSBY, W. W., Chief Engineer, State Roads Commission of Maryland, 2850 N. Charles St., Baltimore, Md..... 1911

LONG, J. J., Instructor in Mechanical Drawing, Brown University, Providence, R. I..... 1911

MANN, F. M., Professor of Architecture and Head of the Architectural Department, University of Illinois, Urbana, Ill..... 1911

* Some of the above are included in the 1911 membership list. A slip with all new names, to be pasted in the membership list, will be sent with the March BULLETIN.

PAINE, E. B., Assistant Professor of Electrical Engineering, University of Illinois, Urbana, Ill..... 1911
RAYMOND, ALFRED, General Manager, Drainage Department, Sewerage and Water Board of New Orleans, New Orleans, La..... 1911
RAYMOND, W. J., Associate Professor of Physics, University of California, Berkeley, Calif..... 1911
RIETZ, H. L., Assistant Professor of Mathematics, University of Illinois, Urbana, Ill..... 1911
VAN BUSKIRK, H. C., Associate Professor of Mathematics, Throop Polytechnic Institute, Pasadena, Calif..... 1911

CHANGES IN ADDRESS.

Mr. R. P. Iyer is now Supervisor, Public Works Department, Parlakemidi P. O., Deccan, India.

Professor R. A. Porter, of Syracuse University, is in Germany this year. His address is Hainholzweg 20, Goettingen.

LIFE MEMBERSHIP.

The election to the Society of two life members, from each of whom we have received $50.00, raises the question whether or not the fees obtained from life memberships should be deposited in a separate fund to the credit of the Society and used as a part of the annual income. The treasurer believes that such fee should be deposited in a savings bank, to the credit of the Society, and the interest used as part of the annual income, or else be allowed to accrue until the fund is sufficient to invest in some good securities. It seems to the treasurer quite feasible to obtain annually a number of life members, so that in the course of time we should have a considerable fund. W. O. W.

OCTAVE CHANUTE.*

Dr. Octave Chanute, who was for many years one of the best-known civil engineers in this country and is noted as a pioneer in aviation, died November 23, 1910, at his home in Chicago, aged nearly 79. Mr. Chanute was born in France in 1832, the son of a professor in the College of France; and came to the United States in 1838 when his father moved to Louisiana to become vice-president of Jefferson College. The family removed to New York City in 1848, and in 1849 Mr. Chanute began his career as an engineer, and was actively engaged in that line for practically sixty years.

The professional career of Mr. Chanute is typical of that of most of the prominent engineers of this country, in that he began at the foot of the ladder. When only seventeen years of age, on introducing himself to the resident engineer in charge of the construction of the Hudson River Railroad, he was told there was "no vacancy"; but he begged for permission to serve gratis "to learn the business," and this being granted after some urging, he considered he had made his fortune when two months later he was made an assistant chainman at $1.12\frac{1}{2}$ a day. However, when he was only twenty-one years old, he was division engineer at Albany, New York, in charge of the terminal facilities and of the maintenance of way between Hudson and Albany.

In 1853 he moved to Illinois, where he worked for ten years; and as chief engineer built portions of several of the great railroads of that section of the country. During this time he was invited to submit designs of the proposed Union Stock Yards at Chicago. His plans were selected in competition with others, and later as chief engineer he built the yards much as they remained to this day. In 1866 he began the first bridge across the Missouri River, which had an ill repute and had been pronounced unbridgeable, but in 1869 Mr. Chanute

* This biographical sketch was prepared by Professor Ira O. Baker.

completed the bridge, a feat which established his reputation among engineers in both this country and in Europe. After this achievement and largely as the result of it, he was placed almost simultaneously in charge of the construction of four railroads in Missouri and Kansas aggregating a total of about 400 miles—a large undertaking in those days.

In 1873 he became chief engineer of the Erie Railroad, a position he held for ten years, during which time the gauge was reduced to the present standard, the line was double-tracked, and the general facilities of the road were greatly improved. For a time he was in temporary charge of the motive and rolling stock in addition to his duties as chief engineer, during which time he improved the locomotives and grades so as to increase the standard through freight train from 18 to 35 cars.

While living in New York the needs of that city for rapid transit attracted Mr. Chanute's attention almost as soon as he came to live in that city. The problem had been discussed for many years without action, when he proposed that it should be investigated by the American Society of Civil Engineers. He was made chairman of the commission which he suggested; and the commission gave five months of careful study to the problem, examining all proposed schemes and conferring with many public bodies and private citizens. The report which was made public in February, 1875, advocated the building of four lines of elevated railroad to be operated by steam locomotives, and proved so convincing that the legislation needed followed almost immediately. The roads were built, and proved a great success from the very first, as everybody knows.

In 1884 he moved to Kansas City, and opened an office as consulting engineer; and in the next few years had general direction of the construction of the bridges on the Chicago, Burlington and Northern Railroad between Savanna and St. Paul, and on the Atchinson, Topeka and Santa Fe Railroad between Chicago and Kansas City.

In 1889 Dr. Chanute moved to Chicago, and continued to

practice as a consulting engineer, but gave considerable time to the preservation of timber and later to the development of aviation. In 1880 he was chairman of a committee appointed by the American Society of Civil Engineers to investigate the subject of timber preservation, and the report of this committee presented in 1885 did much to bring the advantages and economics of timber preservation to the attention of engineers in this country, at a time when but little was known in this country on that subject. The report of this committee was largely his personal work, and was done so well that at its 1910 meeting the society did not consider that there was sufficient additional information available to warrant the appointment of a new committee. For many years Mr. Chanute was recognized as a leading expert on the preservation of timber.

In popular estimation the fame of Dr. Chanute will probably always rest on his work in connection with the development of aërial navigation. As early as 1874 he became interested in the science of aviation; but finding that this "side issue" interfered with his professional work, he laid it aside until 1888 when he was able to give time to it. In 1893 in connection with the Engineering Congress at Chicago he arranged a series of conferences on aërial navigation, and presented two or three papers himself. During the experiments of the Wright Brothers with flying machines from 1901 to 1905 he made at their request numerous calculations for them and supplied them with both information and advice. In 1897 he published a book entitled "Progress in Flying Machines," which contained a very concise and comprehensive account of the science and the state of the art; and in 1909 he presented a paper on "Recent Progress in Aviation" to the Western Society of Engineers. In the summer of 1910 he was awarded a gold medal by the Aëronautical Society of Great Britain. But great as was his work and influence in aviation, Dr. Chanute always regarded it as a "side issue."

Dr. Chanute contributed numerous valuable papers to the American Society of Civil Engineers; and served on several important committees in addition to those mentioned above.

He was president of the American Society of Civil Engineers in 1891, and of the Western Society of Engineers in 1901. While president of the Western society he endowed three medals to be awarded for the best papers in civil, electrical, and mechanical engineering. He was a member of various scientific and engineering societies; and was an honorary member of the Institution of Civil Engineers, and of several other foreign societies. In 1905 the University of Illinois conferred upon him the honorary degree of doctor of engineering.

He was modest and unassuming, courtly in manner, extremely considerate of the feelings and opinions of others, and ever ready to help others, particularly the younger members of the profession, with suggestions or advice.

In 1893 he was chairman of the executive committee of the engineering societies having charge of the International Engineering Congress held in connection with the World's Columbian Exposition in Chicago. At one of the meetings of the chairmen of the various sections, Dr. Chanute inquired of the chairman of the section on engineering education as to progress he was making, and was informed that the chairman was greatly disappointed that the exposition company could not finance the proposed meeting, and would not be able to publish the proceedings—both of which had been promised. Dr. Chanute volunteered to lay the matter before his committee, which agreed to finance the meeting and provide for the publication of the report. The engineering societies financed the meeting, but the organization of the Society for the Promotion of Engineering Education relieved the engineering societies of publishing the proceedings of Section E of the Engineering Congress—Engineering Education. Dr. Chanute's interest at a critical time was invaluable to the Society for the Promotion of Engineering Education. I. O. B.

LETTERS TO THE EDITOR:

EDITOR S. P. E. E. BULLETIN—

Dear Sir: There are very good reasons for not changing the term "polar coördinates" to "vector coördinates," as proposed in the BULLETIN, 1, p. 175.

Thus, the cartesian coördinates are just as much vector coördinates (*i. e.*, coördinates of the vector) as are the polar coördinates. Again, the radius vector, when used as one of the coördinates of a point, is not a vector at all, but a mere scalar.

Yours very truly,
ALEXANDER ZIWET.

ANN ARBOR, MICH.

EDITOR S. P. E. E. BULLETIN—

Dear Sir: I take the liberty of asking you whether you have heard of any institutions that have changed their practice regarding the granting of professional degrees since your report was issued last summer. I am also anxious to know whether any school has yet changed from the C.E. to the B.S. degree. Any information you can give me will be greatly appreciated.

Yours very truly,
FRANK P. MCKIBBEN.

S. BETHLEHEM, PA.

Members of the Society are requested to furnish Professor McKibben with the desired information, preferably through the BULLETIN so that other members may have the benefit.

COLLEGE NEWS.

Colorado Agricultural College.—Professor L. G. Carpenter, head of the department of Civil and Irrigation Engineering, expects in future to give practically all of his time to practical work. He will have offices in Denver and New York City.

Cornell University.—This university has just received a gift of $50,000.00 for a new shop building to be known as the Rand Shops. This building is the first of three which have been reported as necessary to properly care for the expansion of Sibley College. A committee of the trustees has been at work for some time preparing plans for this extension which has become absolutely necessary. The plans for the shops are in preparation and it is expected that the new building will be completed by fall. The present shop building will be occupied for laboratory purposes, the department of experimental engineering having the steam engine laboratory on the first floor and the electrical engineering department, the advanced laboratory on the second floor.

University of Illinois.—Dr. J. A. L. Waddell, the noted bridge engineer of Kansas City, recently delivered two lectures before the faculty and students of the College of Engineering of the University of Illinois, one a technical talk on "Materials of Bridge Engineering and Foundations" and the other a general lecture on bridge construction.—Dr. N. Clifford Ricker, of the Department of Architecture of the University of Illinois, has been appointed a member of a Commission of Awards, to judge and to report upon the relative merits of the designs and plans submitted in competition for three department buildings to be erected in Washington, D. C. The Commission began its sittings on the fourth of January.—The University now leads in the number of members of the S. P. E. E., several having joined since the membership list went to press.

University of Minnesota.—Geo. C. Priester, bachelor of engineering, class 1910, University of Iowa, was recently elected instructor in engineering mathematics in the University of Minnesota and took up his work there at the beginning of the present semester. His application for membership in this Society is already at hand.

Tulane University.—On Thursday, November 17, 1910, the board of administrators of Tulane University of Louisiana, met to consider the proper disposition of a bequest of $60,000, recently received by the University in accordance with a provision of the will of the late Stanley O. Thomas, of New Orleans. After listening to addresses by members of the various faculties and some alumni the board went into executive session and soon announced that a decision had been reached to devote this fund to a new building for the College of Technology, to be known as the Stanley O. Thomas Hall. The additional space will be of great benefit to this college as the number of students has increased about 140 per cent. in the last fifteen years. The laboratory space and the recitation rooms to be provided in the new building are urgently needed at present and they do not provide for many years of growth such as the college has a right to expect. The Department of Electrical Engineering will be housed in the new building, and the space at present occupied by this department will be given up to the Department of Experimental Engineering. The Department of Architecture will also be provided for in the new building.

Union University.—Professor Geo. F. Swain's address delivered at the opening of the new General Engineering Building has been issued as No. 4, Vol. III., of the Union College Bulletin. This puts the address into attractive and useful form for reference.

University of Washington.—The University has recently had the privilege of a visit from Professor Jacoby, of Cornell, who is on a half-year's leave of absence for the purpose of getting in touch with the latest practice in bridge engineering throughout the country. Professor Jacoby found here three

former students, A. H. Fuller, '98, C. C. More, '99, and C. W. Harris, '05, and a former colleague, E. J. McCaustland, all of whom are now connected with the civil engineering staff. He delivered an address here before the senior and junior civil engineers, which was enjoyed and appreciated by all.

West Virginia University.—Dr. D. B. Purinton, President of the University for the past ten years, recently tendered his resignation to take effect at the close of this college year. President Purinton will retain a professorship in the University, having voluntarily relinquished the duties of administration, which he thought should be entrusted to a younger man.—Dr. Thomas E. Hodges has been selected as President Purinton's successor. Dr. Hodges was recently Professor of Physics in the University and is now minority member of the State Board of Control. The new president is without doubt the most popular man in the State. He is a broad, liberal scholar and business man as well as a scientist. It is believed that our engineering departments will be greatly strengthened by the influence of Dr. Hodges, who was for many years our teacher of engineering physics.—A chapter of the Phi Beta Kappa Society was installed at the University on December fifth. The University was granted a charter for the chapter by the unanimous vote of the Senate and the almost unanimous consent of the Council of the Society at its meeting in New York last September.

The following report will be presented at the Pittsburgh meeting of the Society, to be held in June. Correspondence regarding this report should be addressed to Professor E. V. Huntington, 27 Everett St., Cambridge, Mass.

PRELIMINARY REPORT OF THE COMMITTEE ON THE TEACHING OF MATHEMATICS TO STUDENTS OF ENGINEERING.

To the Society for the Promotion of Engineering Education:

The committee was appointed at a joint meeting of mathematicians and engineers held in Chicago, December 30–31, 1907, under the auspices of the Chicago Section of the American Mathematical Society, and Sections A and D of the American Association for the Advancement of Science,* and on the suggestion of officers of the Society for the Promotion of Engineering Education who were there present, the committee was instructed to report to this Society.

The membership of the committee is as follows:

ALGER, Philip R., professor of mathematics, U. S. Navy, Annapolis, Md.

CAMPBELL, Donald F., professor of mathematics, Armour Institute of Technology, Chicago, Ill.

ENGLER, Edmund A., president of the Worcester Polytechnic Institute, Worcester, Mass.

HASKINS, Charles N., assistant professor of mathematics, Dartmouth College, Hanover, N. H.

HOWE, Charles S., president, Case School of Applied Science, Cleveland, Ohio.

KUICHLING, Emil, consulting civil engineer, New York City.

MAGRUDER, William T., professor of mechanical engineering, Ohio State University, Columbus, Ohio.

MODJESKI, Ralph, civil engineer, Chicago, Ill.

* For an account of the Chicago meeting, see *Science* for 1908 (July 12, 24, and 31; August 7 and 28; and September 4).

OSGOOD, William F., professor of mathematics, Harvard University, Cambridge, Mass.

SLICHTER, Charles S., consulting engineer of the U. S. Reclamation Service, professor of applied mathematics, University of Wisconsin, Madison, Wis.

STEINMETZ, Charles P., consulting engineer of the General Electric Company, professor of electrical engineering, Union University, Schenectady, N. Y.

SWAIN, George F., consulting engineer, professor of civil engineering, Harvard University, Cambridge, Mass.

TOWNSEND, Edgar J., dean of the College of Science and professor of mathematics, University of Illinois, Urbana, Ill.

TURNEAURE, Frederick E., dean of the College of Mechanics and Engineering, University of Wisconsin, Madison, Wis.

WALDO, Clarence A., head professor of mathematics, Washington University, St. Louis, Mo.

WILLIAMS, Gardner S., consulting engineer, professor of civil, hydraulic and sanitary engineering, University of Michigan, Ann Arbor, Mich.

WOODWARD, Calvin M., dean of the School of Engineering and Architecture and professor of mathematics and applied mechanics, Washington University, St. Louis, Mo.

WOODWARD, Robert S., president of the Carnegie Institution of Washington, Washington, D. C.

ZIWET, Alexander, professor of mathematics, University of Michigan, Ann Arbor, Mich.

HUNTINGTON, Edward V., chairman, assistant professor of mathematics, Harvard University, Cambridge, Mass.

After deliberation, the committee decided that it could best carry out the purpose for which it was appointed by preparing *a synopsis of those fundamental principles and methods of mathematics which, in the opinion of the committee, should constitute the minimum mathematical equipment of the student of engineering*, and the chairman was instructed to submit for discussion a preliminary draft of such a synopsis. In order to secure as wide a discussion as possible, the committee voted to accept the invitation of the Society for the Promotion

of Engineering Education to publish this preliminary draft in the BULLETIN in advance of its presentation to the Society, and to solicit criticisms and suggestion from all members of the Society, whether members of the committee or not.

It should be distinctly understood that this advance publication is merely a tentative draft, prepared by the chairman, and still under discussion by the members of the committee; *the committee as a whole is not to be held responsible for any part of the synopsis in its present form.*

The synopsis, as now planned, will consist of six parts:*

1. A Syllabus of Elementary Algebra;
2. A Syllabus of Plane Trigonometry;
3. A Syllabus of Differential and Integral Calculus;
4. Numerical Computation and the Solution of Equations;
5. Geometry (Elementary and Analytical);
6. A Syllabus of Elementary Dynamics;

preceded by the following introductory note, explaining its scope and purpose:

"It is hoped that this report may be serviceable in two ways: first, to the teacher, as an indication of where the emphasis should be laid; and secondly, to the student, as a syllabus of facts and methods which are to be his working tools. It does not include data for which the student would properly refer to an engineers' hand-book; it includes rather just those things for which he ought never to be obliged to refer to any book—the things which he should have constantly at his fingers' ends.

"The teacher of mathematics should see to it that at least these facts are perfectly familiar to all his students, so that the professor of engineering may presuppose, with confidence, at least this much mathematical knowledge on the part of his students. On the other hand, if the professor of engineering needs to use, at any point, more advanced mathematical methods than those here mentioned, he should be careful to explain them to his class.

* The first two parts appear in the present issue.

"The committee has not found it possible to propose a detailed course of study. The order in which these topics should be taken up must be left largely to the discretion of the individual teacher. The committee is firmly of the opinion, however, that whatever order is adopted, the principal part of the course should be *problems worked by the students,* and that all these problems should be *solved on the basis of a small number of fundamental principles and methods, such as are here suggested.*

"The defects in the mathematical training of the student of engineering appear to be largely in knowledge and grasp of fundamental principles, and the constant effort of the teacher should be to ground the student thoroughly in these fundamentals, which are too often lost sight of in a mass of details.

"A pressing need at the present time is a series of synoptical text-books, which shall present, (1) the fundamental principles of the science in compact form, and (2) a classified and graded collection of problems (which would naturally be subject to continual change and expansion). It is the hope of the committee that this report, which is confined to the first part of the desired text-book, will stimulate throughout the country practical contributions toward the second."

Criticisms and suggestions from any source are invited, and may either be offered to the BULLETIN *for publication, or be sent directly to the Chairman, Professor Edward V. Huntington, 27 Everett Street, Cambridge, Mass.*

In the early part of its investigation the committee collected a large amount of information in regard to the present status of mathematical instruction for engineering students. Since that time, however, a much more inclusive inquiry has been undertaken by the International Commission on the Teaching of Mathematics, of which the American Commissioners are Professors D. E. Smith, J. W. A. Young and W. F. Osgood. In order to avoid unnecessary duplication, this committee voted to turn over all the results of its own inquiry in this field

to the larger commission, to be worked up in accordance with the general scheme adopted by that commission, and to be incorporated in their report. This material is therefore not included in the present report.

Respectfully submitted,

EDWARD V. HUNTINGTON,

Chairman.

A SYLLABUS OF THE FORMAL PART OF ELEMENTARY ALGEBRA.

This syllabus is intended to include those facts and methods of elementary algebra which a student who has completed a course in that subject should be expected to "know by heart"—that is, those fundamental principles which he ought to have made so completely a permanent part of his mental equipment that he will never need to "look them up in a book."

It is not intended as a program of study for beginners, and no attempt has been made to arrange the topics in the order in which they should be taught. In reviewing the subject, however, either at the end of the course in algebra, or at the beginning of any later course, such a syllabus will be found serviceable to both teacher and student; and in the hands of a skillful teacher, and supplemented by an adequate collection of problems, it might well be made the basis of a course of study conducted by the "syllabus method."

One of the chief defects in the present-day teaching of algebra is the multiplicity of detached rules with which the student's mind is burdened;* and every successful attempt to knit together a number of these detached rules into a single general principle (provided this principle is simple and easily applied) should conduce to economy of mental effort, and diminish the liability to error.

TABLE OF CONTENTS.

I. TRANSFORMATION OF ALGEBRAIC EXPRESSIONS.
- General laws of addition and multiplication.
- Type-forms of multiplication (Factoring).
- Fractions.
- Negatives.
- Radicals and Imaginaries.
- Exponents and Logarithms.

II. SOLUTION OF EQUATIONS.
- Legitimate operations on equations.
- To solve a single equation.
  - Quadratic equations.
  - Exponential equations.
- To solve a set of simultaneous equations.

* For example, in a recent prominent text-book there are no less than *fifty* italicized rules in the part of the book preceding quadratic equations!

III. MISCELLANEOUS TOPICS.
Ratio and proportion.
Variation.
Inequalities.
Arithmetical, geometric, and harmonic progressions.

I. TRANSFORMATION OF ALGEBRAIC EXPRESSIONS.

The ordinary operations of transforming and simplifying algebraic expressions should be so familiar to the student that he performs them almost instinctively; at the same time he should be able, whenever called upon, to justify each step of his work by reference to some one or more of a small number of well established principles.

For example, if the student is asked *by what authority* he replaces $\frac{a+x}{b+x}$ by $\frac{a}{b}$, or $\sqrt{a^2+b^2}$ by $a+b$ (to mention only two of the commonest blunders), he will be forced to recognize either that he is making use of methods that he has never proved, and that are in fact erroneous, or else (which is more likely) that he is working altogether in the dark, without any conscious reason for the steps he has taken.

The following list of such principles, while making no pretense at logical completeness, will be sufficient for all practical purposes.

General laws of addition and multiplication.

$a+b=b+a.$ $\qquad ab=ba.$ (Commutative laws.)

$(a+b)+c=a+(b+c).$ $\qquad (ab)c=a(bc).$ (Associative laws.)

$a(b+c)=ab+ac.$ (Distributive law.)

$a+0=a.$ $\qquad a\times 1=a.$ $\qquad a\times 0=0.$

These laws hold when a, b, c are any of the quantities that occur in ordinary algebra, whether "real" or "complex."* *The student should be constantly encouraged to test general algebraic statements by substituting concrete numerical values.*

Type-forms of multiplication (Factoring).

The following type-forms of multiplication are the ones that are most important to remember:

* This syllabus is confined chiefly to the algebra of real quantities; the algebra of complex quantities will be treated only incidentally.

$$a^2 - b^2 = (a-b)(a+b),$$

$$a^3 - b^3 = (a-b)(a^2+ab+b^2),$$

and so on; the general case is best remembered in the form

$$1 - x^n = (1-x)(1+x+x^2+x^3+\cdots+x^{n-1}).$$

Note also that in the algebra of real quantities, $a^n + b^n$ is divisible by $a+b$ when and only when n is *odd*. Thus:

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2).$$

Further: $$(x+a)(x+b) = x^2 + (a+b)x + ab,$$

and the "binomial theorem":

$$(a+b)^2 = a^2 + 2ab + b^2, \quad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3,$$

$$(a-b)^2 = a^2 - 2ab + b^2, \quad (a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3;$$

$$(a+b)^n =$$

$$a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 + \cdots$$

$$\cdots + \frac{n(n-1)(n-2)}{3!}a^3b^{n-3} + \frac{n(n-1)}{2!}a^2b^{n-2} + nab^{n-1} + b^n,$$

where $k! =$ "k factorial" $= 1 \times 2 \times 3 \times \cdots \times k$.

Fractions.

Def. If $bx = a$, then and only then we write $x = \frac{a}{b}$ (or a/b, or $a \div b$).

Here a is called the numerator and b the denominator of the fraction. A fraction with a zero denominator, as $a/0$, does not represent any definite quantity. For, if a is not zero, there is *no* quantity x such that $0 \times x = a$; and if $a = 0$, then *every* quantity x will have this property. Hence, *the denominator of a fraction must always be different from zero.*

From the definition, $a/1 = a$; also

$$\frac{a}{a} = 1, \quad \frac{0}{a} = 0, \qquad (a \neq 0).^*$$

* The symbol $\neq$ means "not equal to."

To add two fractions with common denominator:

$$\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}.$$

To multiply two fractions:

$$\frac{a}{b}\times\frac{x}{y}=\frac{ax}{by}.$$

To divide by a fraction, "invert the divisor and multiply":

$$\frac{a}{b}\div\frac{x}{y}=\frac{a}{b}\times\frac{y}{x}=\frac{ay}{bx}.$$

The value of a fraction is not changed if we multiply (or divide) both the numerator and the denominator by any quantity not zero:

$$\frac{a}{b}=\frac{ma}{mb} \qquad (m\neq 0).$$

This is the most important principle concerning fractions.

For example, *to reduce two fractions to a common denominator,* we have merely to multiply numerator and denominator of each fraction by a suitable factor.

Again, *to simplify a complex fraction,* we multiply the whole numerator and the whole denominator by any quantity which will "absorb" all the subsidiary denominators. Thus, by multiplying by xyz, we have

$$\frac{\frac{a}{x}+\frac{b}{y}}{\frac{c+d}{z}}=\frac{ayz+bxz}{(c+d)xy},$$

at once, by a single mental process. (The common practice of reducing the numerator and denominator separately, and then inverting the denominator and multiplying, is tedious and clumsy.)

Def. If $bx=1$, then $x=1/b$, which is called the *reciprocal* of b. To divide by $b\,(b\neq 0)$ is the same as to multiply by the reciprocal of b.

Negatives.

Def. If $a+x=0$, then and only then we write $x=-a$. In particular, $-(-a)=a$.

If a is not zero, $-a$ is always *opposite* to a; that is, if a is positive, $-a$ is negative, and if a is negative, $-a$ is positive.

Thus, if $a=-3$, which is a negative quantity, then $-a=3$, which is positive.

The notation $|a|$, which is coming into use more and more widely, means the *absolute value of a*, that is, the numerical value of a regardless of sign; thus, $|5|=5$, $|-5|=5$.

The laws of operation with the minus sign are best remembered by regarding $-a$ as the product of a and -1:

$$(-1)\times a=-a,$$

whence, in particular (putting $a=-1$),

$$(-1)\times(-1)=1.$$

When this is done, the customary formulas:

$$(-a)(-b)=ab,\quad (-a)(b)=-ab,\quad \frac{-a}{b}=\frac{a}{-b}=-\frac{a}{b},\quad \frac{-a}{-b}=\frac{a}{b},$$

become immediate consequences of the general laws of multiplication and division, and therefore need not be separately memorized; and the same is true of the formula

$$a-(x+y-z)=a-x-y+z,$$

which, when remembered in the following form, becomes an immediate application of the distributive law: "a minus sign in front of a parenthesis must be 'distributed' through every term within, if the parentheses are to be taken away."

By knitting together in this way the rules for negatives with the general rules of operation, the total number of processes to be remembered and applied, and hence the liability to error, is materially reduced.

Def. If $a+x=b$, then and only then we write $x=b-a$. It is easily shown that $b-a=b+(-a)$; that is, *subtracting any quantity a is the same as adding the opposite of a.*

Radicals.

Def. If a is *positive*, and n is any positive integer, there will always be one positive value of x such that $x^n=a$. This value x is denoted by $\sqrt[n]{a}$, and is called the (principal) *nth root of a.*

It should be noticed that while there are (for example) two square roots of 9, namely 3 and -3, it is only the positive one of these two values that is denoted by $\sqrt{9}$; that is, the mark $\sqrt{9}$ means 3 and not -3.

The radical sign, except in the case of square roots, and sometimes in the case of cube roots, should always be replaced by *fractional exponents* (see below) when it is desired to compute with these quantities; this done, no special rules for the manipulation of radicals need then be remembered beyond the general laws of exponents.

Square roots. If a and b are positive,

$$\sqrt{a^2b} = a\sqrt{b}, \quad \text{and} \quad \sqrt{a}\sqrt{b} = \sqrt{ab}.$$

Note also the process called "rationalizing the denominator (or numerator) of a fraction"; for example,

$$\frac{c}{\sqrt{a}+\sqrt{b}} = \frac{c}{\sqrt{a}+\sqrt{b}} \times \frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \frac{c(\sqrt{a}-\sqrt{b})}{a-b};$$

$$\frac{\sqrt{1-x}}{\sqrt{1+x}} = \frac{\sqrt{1-x}}{\sqrt{1+x}} \times \frac{\sqrt{1-x}}{\sqrt{1-x}} = \frac{1-x}{\sqrt{1-x^2}}.$$

Def. If a is *negative*, and n is *odd*, there will always be one negative value of x such that $x^n = a$; this value is denoted by $\sqrt[n]{a}$, and is called the (principal) nth root of a.

Thus $\sqrt[3]{-8} = -2.$

Imaginaries.

If a is *negative*, and n is *even*, then there is *no* positive or negative *nth* root of a. Hence, such quantities do not occur in the algebra of positive and negative quantities. They occur only in the more general *algebra of complex quantities;* in this algebra every quantity a (except zero) has n *distinct nth roots*, the notation $\sqrt[n]{a}$ being applied, as occasion requires, to any one of these n values. The detailed study of this general algebra is probably too difficult for a first course; for such applications as occur in elementary work, the following working rules are sufficient:

1) In manipulating a complex quantity of the form $\sqrt{-b}$, where b is positive, write $\sqrt{-b} = \sqrt{-1}\sqrt{b} = i\sqrt{b}$, and treat i like any other letter; then simplify the result by the relation $i^2 = -1$.

2) Every complex quantity can be written in the form $a + ib$, where a and b are "real" (that is, positive, negative, or zero); and if $a + ib = a' + ib'$, then $a = a'$ and $b = b'$.

In electrical engineering the letter i is used to denote current, and $\sqrt{-1}$ is denoted by j.

Exponents.

The subject of negative and fractional exponents is a part of algebra in which the preparation of the student is apt to be especially unsatisfactory.

Definition of negative and fractional exponents. If a is positive, and p and q are any positive integers, then

$$a^0 = 1, \qquad a^{-p} = \frac{1}{a^p}, \qquad a^{1/q} = \sqrt[q]{a}, \qquad a^{p/q} = \sqrt[q]{a^p}.$$

Laws of operation with exponents.

If a and b are positive, then:

$$a^{m+n} = a^m a^n, \quad a^{mn} = (a^m)^n, \quad (ab)^m = a^m b^m,$$

$$a^0 = 1, \qquad a^{-m} = \frac{1}{a^m}, \qquad a^{1/m} = \sqrt[m]{a}.$$

All these laws hold for any values of m and n; the three fundamental ones can readily be recalled to mind through simple special cases, such as a^3a^2, $(a^3)^2$, and $(ab)^3$.

The three other laws commonly mentioned, namely

$$a^{m-n} = a^m/a^n, \quad a^{m/n} = \sqrt[n]{a^m}, \quad (a/b)^m = a^m/b^m,$$

are immediate corollaries of those just mentioned.

If a is *negative*, and m not an integer, a^m will, in general, be a complex quantity. In such cases, let $a' = -a$, so that a' is positive, and write $a^m = (-1)^m a'^m$, where $(-1)^m$ must then be handled according to the rules of operation in the algebra of complex quantities.

Logarithms.

The subject of logarithms should be taught in logical connection with the subject of exponents. The common practice of separating these subjects, and treating logarithms as a part of trigonometry, is unfortunate. Numerous applications of logarithms can be found that have nothing to do with trigonometry; moreover, the training in the use of logarithms which a student gets in trigonometry is usually quite inadequate as a preparation for the applications of logarithms in any of his later work outside of surveying.

Def. The *logarithm* of a (positive) number, to any (positive) base, is the *exponent* of the power to which the base must be raised to produce that number.

Thus, the notation

$$x = \log_b N$$

means

$$b^x = N.$$

Note that negative numbers in general have no logarithms in the algebra of real quantities.

From the laws of exponents we have, whatever the base may be:

$$\log (ab) = \log a + \log b, \quad \log \left(\frac{a}{b}\right) = \log a - \log b,$$

$$\log (a^n) = n \log a, \qquad \log \sqrt[n]{a} = \frac{1}{n} \log a,$$

$$\log 1 = 0, \quad \log (base) = 1.$$

Only two bases are in common use. For purposes of numerical computation, the base chosen is 10, and in this system $\log (10^n) = n$.

(See chapter on numerical computation.) In higher mathematics, the base $e = 2.718 \cdots$ is used, for the reason that the use of this base simplifies certain formulas in the calculus; in this system $\log (e^n) = n$.

Change of base. To find $\log_e N$ when $\log_{10} N$ is known, let $x = \log_e N$, that is, $e^x = N$. Then take the logarithm of both sides of this equation to base 10, and solve for x.

The resulting formula, $\log_e N = (\log_{10} N)/(\log_{10} e)$, is so easily obtained in this way that it is not worth while to remember it separately. The approximate values

$$\log_{10} e = .4343, \text{ and } \log_e N = (2.3026) \log_{10} N,$$

however, are useful to remember.

II. Solution of Equations.

Legitimate operations on equations. If a given equation is true, it will still be true if we

(*a*) *add* any quantity we please to both sides;

(*b*) *subtract* any quantity we please from both sides;

(*c*) *multiply* both sides by any quantity we please;

(*d*) *divide* both sides by any quantity we please *except zero*,

(*e*) raise both sides to any *positive integral power;*

(*f*) *extract any *positive integral root* of both sides, except that if an *even* root is extracted, the double sign $\pm$ must be used;

(*g*) *take the *logarithm* of both sides (provided both sides are positive).

In regard to (*d*), we must never divide both sides by an unknown quantity without first excluding the possibility that that quantity is zero.

In (*f*), the restriction stated means, for example, that from $A^2 = B$ we can infer merely that $A = \pm \sqrt{B}$; that is, that *either* $A = \sqrt{B}$, or $A = -\sqrt{B}$; but we cannot tell which.

To solve a single equation in x, means to find all the values of x that satisfy the equation, or to show that none such exist.

Any value of x that satisfies the equation is called a *root* of the equation.

In testing a root, the only safe method is to substitute the given value in each side of the equation separately, and see whether the results, when reduced, are equal. Thus, we should find that $x = -2$ is a root of the equation $x = 2 - \sqrt{12 - 2x}$, and that $x = 4$ is not a root.

In this connection it should be noticed that if we square both sides of a given equation, the new equation will, in general, have more roots than the given equation. Thus (to use the same example), by squaring $x - 2 = -\sqrt{12 - 2x}$ we have $x^2 - 2x - 8 = 0$. This equation has of course the root -2, since $x = -2$ satisfies the original equation from which this was derived; but it has also the root 4, which was not a root of the original equation.

The formal process usually called "solving the equation" means merely transforming the equation, by a judicious choice of the legitimate operations, into a form in which the solutions are obvious.

If this is not possible, we must have recourse to the *method of trial and error* which, while often laborious, is always applicable in numerical cases. (See chapter on numerical computation.)

If an equation is given in the factored form:

$$(x - \alpha)(x - \beta)(x - \gamma) \cdots = 0,$$

* In the algebra of complex quantities (*f*) and (*g*) are not applicable.

then the roots are obviously $x=\alpha$, $x=\beta$, $x=\gamma, \cdots$. Thus, the roots of $x(x+2)=0$ are 0 and -2.

Quadratic equations. To solve the quadratic equation

$$ax^2+bx+c=0,$$

we may divide through by a:

$$x^2+\frac{b}{a}x \qquad = \qquad -\frac{c}{a}$$

and then "complete the square":

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2};$$

whence,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a};$$

or, we may use the general result just obtained as a formula.

The quantity which must be added to both sides in "completing the square" is obvious by analogy with $x^2+2mx+m^2$, so that this method requires less effort of the memory than the method of solution by formula.

The "method of factoring" is very convenient in certain special cases, when the factors can be obtained by inspection.

The method still sometimes used, of first multiplying through by $4a$ to avoid fractions, is apt to lead to confusion, and should be discouraged.

From the formula it is evident that the *sum of the roots* is $x_1+x_2=-b/a$, and the *product of the roots* is $x_1x_2=c/a$; also, if the coefficients, a, b, c, are real, the roots will be real-and-distinct, real-and-coincident, or imaginary, according as b^2-4ac is positive, zero, or negative.

Exponential equations. To solve an equation of the form $a^x=b$, when a and b are positive, *take the logarithm of both sides:* $x \log a=\log b$; and then solve for x.

To solve a set of simultaneous equation in x, y, z $\cdots$ means to find all the *sets of values* of $x, y, z, \cdots$, that satisfy all the equations at once, or show that none such exist.

Two simultaneous equations of the first degree, as $ax+by=c$ and $Ax+By=C$, can always be solved in a couple of lines, if the work is arranged as follows:

$$\begin{array}{rr|rr} 7x-6y=1 & & -5 & -2 \\ 14x-10y=3 & & 3 & 1 \\ \hline (-35+42)x=-5+9 & & & \\ (12-10)y=-2+3 & & & \end{array}$$

whence the values of x and y are obvious, provided $aB-bA$ *is not zero.* (If $aB-bA=0$, there is either no pair of values x, y that satisfies both the equations, or else there are an infinite number of pairs of values that do so; in this latter case, the equations are not independent, that is, either of them can be derived from the other.)

The theory of simultaneous equations, and sometimes the numerical computation, is facilitated by the use of determinants.

In general, n independent equations will suffice to determine n unknown quantities.

III. Miscellaneous Topics.

Ratio and Proportion. The "ratio of a to b" means simply the fraction a/b; and a "proportion" is simply an equation between two ratios.

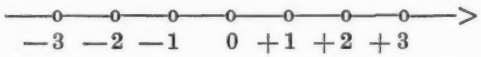
The notation $a:b::c:d$ should be replaced by the equation $a/b=c/d$; and all special terminology, such as "taking a proportion by alternation," "by composition," etc., should be dropped in favor of the ordinary language of the equation.

Variation. The statement "y varies as x," or "y varies directly as x," or "y is proportional to x," means $y=kx$, where k is some constant. Similarly, "y varies inversely as x," means $y=k/x$; "y varies inversely as the square of x," means $y=k/x^2$. The constant k can always be determined if we know any pair of values of x and y that belong together.

The statement "y varies as u *and* v," means y varies as the *product* of u and v, that is, $y=kuv$.

Inequalities. The notions of "greater and less" are thoroughly familiar when we are dealing only with positive quantities, but the extension of these terms to the algebra of all real quantities (positive, negative, and zero) is apt to cause some confusion.

(a) All *real* quantities (positive, negative, and zero) may be represented by the points of a directed line (running, say, from left to right):



and the notation $a<b$ (read: "a algebraically less than b") means simply that a *precedes* b, or a lies on the left of b, along this line.

Similarly, $a>b$ (read: "a algebraically greater than b") means that a *comes after* b, or lies on the right of b, along the line. (The idea that a negative quantity is a magnitude whose *size* is in some way "less than nothing" should be carefully avoided.)

Obviously, if a and b are any real quantities, one and only one of the three relations: $a=b$, $a<b$, and $a>b$, will hold between them; further, if $a<b$ and $b<c$, then $a<c$.

(*b*) *Complex* quantities require for their representation the points of a plane instead of the points of a line, and the symbols $<$ and $>$ are not used in connection with these quantities.

Legitimate operations on inequalities. If a given inequality is true, it will still be true if we

(*a*) *add* any quantity we please to both sides;

(*b*) *subtract* any quantity we please from both sides;

(*c*) *multiply* both sides by any *positive* quantity;

(*d*) *divide* both sides by any *positive* quantity;

(*e*) raise both sides to any *positive power* (integral or fractional), *provided both sides are positive.*

(*f*) take the *logarithm* of both sides, *provided both sides are positive.*

If we multiply or divide both sides by any *negative* number, we must *reverse the sense of the inequality.*

The neglect of the rules for handling inequalities is the source of many common errors.

Arithmetical Progression.

In an arithmetical progression:

$$a, \quad a+d, \quad a+2d, \quad a+3d, \quad \cdots,$$

each term is obtained from the preceding by *adding* a constant quantity.

The nth term is obviously $l = a + (n-1)d$.

The sum of n terms is $S = \frac{a+l}{2} n$.

This formula is most easily remembered in the form:

$S =$ (average of the first and last terms) $\times$ (number of terms).

The *arithmetic mean* between a and b is $A = \frac{1}{2}(a+b)$.

Geometric Progression.

In a geometric progression:

$$a, \quad ar, \quad ar^2, \quad ar^3, \quad \cdots,$$

each term is obtained from the preceding by *multiplying* by a constant quantity.

The nth term is obviously $l = ar^{n-1}$.

The sum of n terms is $S = \frac{a(1-r^n)}{1-r}$.

This formula is best remembered in connection with the rule for factoring:

$$1 - r^n = (1-r)(1 + r + r^2 + r^3 + \cdots + r^{n-1}).$$

The *geometric mean* between a and b is $G = \sqrt{ab}$.

The geometric mean is also called the *mean proportional.*

Infinite geometric progression. If $|r| < 1$, the sum of n terms approaches the limit

$$\frac{a}{1-r}$$

as n increases indefinitely (since, in the expression for S, if $|r| < 1$, r^n approaches zero).

Harmonic Progression.

A harmonic progression is a series of terms whose reciprocals are in arithmetical progression. (The harmonic progression is not of great importance.)

The *harmonic mean* between a and b is $H = \frac{2ab}{a+b}$.

A SYLLABUS OF PLANE TRIGONOMETRY.

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CHAPTER I.

Sine, Cosine, and Tangent of Acute Angles.

1. *Definition of sine, cosine, and tangent of an acute angle* x.—In any right triangle, if x is one of the acute angles, the sine, cosine and tangent of x are defined as ratios between the sides of the triangles, as follows:

$$\sin x = \frac{\text{side opp.}}{\text{hypot.}} \qquad \cos x = \frac{\text{side adj.}}{\text{hypot.}}$$

$$\tan x = \frac{\text{side opp.}}{\text{side adj.}}$$

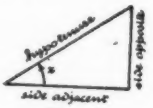


Fig. 1.

These ratios are pure numbers, depending only on the size of the angle.

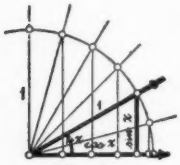


Fig. 2.

2. To trace the changes in these numbers when the angle changes from 0° to 90°, draw the figure so that the denominator of the ratio is kept constant, say equal to 1 inch, and trace the changes in the numerator. Thus, from Fig. 2, when x goes from 0° to 90°, sin x goes from 0 to 1, and cos x goes from 1 to 0; from Fig 3, when x goes from 0° to 90°, tan x goes from 0 to infinity.

Fig. 3.

3. *Tables.*—The ratios thus defined are called "trigonometric functions" of the angle, and their values have been tabulated, to 4, 5 or 6 places of decimals, in the "tables of trigonometric functions." Before using the printed tables, the student should make his own table, for a few angles, by graphical construction, with a protractor, to two places of decimals.*

* It is clear from the figure that the values of cos x from 0° to 90° are the same as the values of sin x in reverse order; note how this fact is made use of to save space in the tables.

4. The functions of 30°, 45°, and 60° can be found exactly, without the use of the table. Thus, in the triangles which occur in Fig. 4, it is readily proved by the Pythagorean Theorem that if the hypotenuse is 1 inch, the shortest side is $\frac{1}{2}$ in., the longest side is $\frac{1}{2}\sqrt{3}$ in., and the middle-sized side $\frac{1}{2}\sqrt{2}$ in. Hence any function of 30°, 45°, or 60° can be read off the figure by inspection. For example,

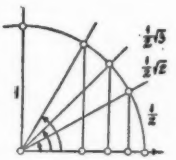


FIG. 4.

$$\sin 30° = \tfrac{1}{2}, \quad \tan 45° = 1, \quad \tan 60° = \sqrt{3}; \quad \text{etc.}$$

5. It is frequently required *to find the remaining functions of an angle when any one function is given.* To do this, draw a right triangle, mark one of the angles, and mark two sides to correspond to the given function. Then compute the remaining side by the Pythagorean Theorem, and read off any desired function from the completed figure. For example,

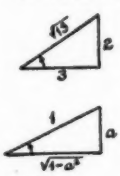


FIG. 5.

Given, $\tan x = \frac{2}{3}$. From the figure, $\sin x = 2/\sqrt{13}$; etc.

Given, $\sin x = a$. From the figure, $\tan x = a/\sqrt{1-a^2}$; etc.

To construct an angle when any one of its functions is given, first find the *tangent* of the angle; when the tangent is known, the construction of the angle is obvious.

6. The notation $\sin^2 x$, etc., is used as an abbreviation for $(\sin x)^2$; etc.

The following fundamental relations are easily proved and remembered from the figure: for any angle x,

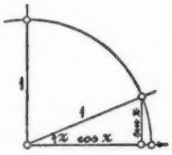


FIG. 7.

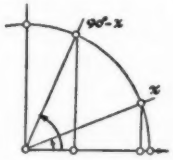


FIG. 8.

$$\sin^2 x + \cos^2 x = 1, \quad \tan x = \frac{\sin x}{\cos x}, \quad \begin{matrix} \sin (90° - x) = \cos x. \\ \cos (90° - x) = \sin x. \end{matrix}$$

7. The student should be thoroughly drilled in the definitions of the sine, cosine and tangent, in right triangles in all possible positions in the plane regardless of lettering. Thus, the mental process should be as follows: pointing at the figure, "the tangent of *this* angle is *this* side, divided by *this* side"; etc.

FIG. 9.

The following forms of the original equations are especially useful, and should be emphasized:

$$\text{side } opp. = \text{hypot.} \times sine; \quad \text{side } adj. = \text{hypot.} \times cosine.$$

SOLUTION OF RIGHT TRIANGLES.

8. We recall that in any right triangle, the sum of the squares on the two legs is equal to the square on the hypothenuse, and the sum of the acute angles is 90°. Hence, when either acute angle is known, the other may be found; and the sine of either acute angle is the cosine of the other.

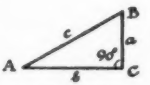


FIG. 10.

$$c^2 = a^2 + b^2, \quad \sin A = \cos B.$$

9. By the aid of a table of sines, cosines and tangents, *when any two parts of a right triangle, besides the right angle, are given, the remaining parts may be found* (except in the case where the given parts are the two acute angles, in which case the triangle is not determined).

For, we have merely to remember the definitions of the functions, selecting the equations so that only one unknown appears in each equation; then solve for the unknown quantity, and compute by the aid of the tables. The results should be checked by substituting in some relation not used in the direct computation.*

* This computation, like many other numerical computations, can often be shortened by the use of the slide rule, or by the use of logarithms; in fact, tables are provided which give the logarithms of the trigonometric functions directly in terms of the angles; but the student should thoroughly understand the use of the functions themselves before he begins to use the logarithmic tables.

10. Numerous problems involving right triangles: isosceles triangles, polygons, oblique triangles solved by means of right triangles, heights and distances, surveying problems, etc.

ORTHOGONAL PROJECTION. COMPONENTS OF FORCES, ETC.

11. The *projection of a length AB* on any line is *the given length times the cosine of the angle* between the lines. (Proof from the definition of cosine.)

The *projection of a plane area* upon any fixed plane is *the given area times the cosine of the angle* between the planes. Proof by the theorem of limits.)

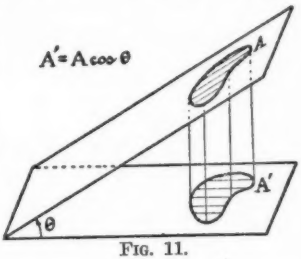


FIG. 11.

12. *The component of a force* along any fixed axis is the *magnitude of the force times the cosine of the angle* between the force and the axis.

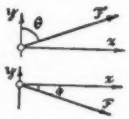


FIG. 12. 13.

Since we usually require the components along two rectangular axes, it is important to remember that $\cos(90° - x) = \sin x$. The mental process should be as follows:

In Fig. 12, the component of F along the y-axis is F times the cosine of θ; the component of F along the x-axis is F times the cosine of the other angle, which is F times the *sine* of θ; that is, $F_y = F\cos\theta$; $F_x = F\sin\theta$. Similarly, in Fig. 13, $F_x = F\cos\phi$; $F_y = -F\sin\phi$ (minus, because it pulls backward along that line).

The components of velocities, accelerations, or any other vector quantities are to be handled in the same way.

13. Every problem should be accompanied by a *sketch* or *diagram*, to show that the student understands the meaning of each step of his work. And in many cases, an accurate graphical solution on a drawing board may be used as a valuable check on the correctness of the numerical computation.

14. Note. *That portion of trigonometry which has been outlined up to this point is so elementary in character, and so readily understood and appreciated by the student, that it may well be introduced much earlier in the course than is usually done—perhaps even as early as the elementary course in plane geometry.*

CHAPTER II.

The Trigonometric Functions of Any Angle.

15. *Angles in general.*—An angle, as the term is used in applied mathematics, is the amount of rotation of a moving radius OP about a fixed point O, measured from a fixed line

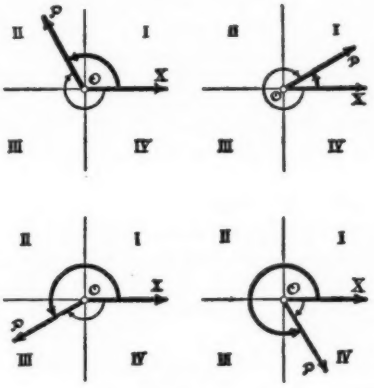


Fig. 14.

OX. Here OX is called the *initial line* and OP the *terminal line* of the angle. Counterclockwise rotation is positive, and angles are added and subtracted as algebraic quantities. The quadrants are numbered as in the figure; an "angle in quadrant *II*" for example, means an angle whose terminal line lies in quadrant *II*.

16. *Congruent angles* are angles differing by any multiple of 360°.

17. *Complementary angles* are angles whose sum is 90°; *supplementary angles* are angles whose sum is 180°.

18. Units of angular measurement are: the *degree*, subdivided into minutes and seconds, or decimally; the *grade*,

subdivided decimally; and the *radian,* subdivided decimally.

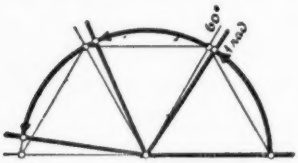


FIG. 15.

1 degree = 1° = 1/90th of a right angle;

1 grade = 1/100th of a right angle (used in France);

1 radian = angle subtended by an arc equal to the radius.

Since ratio of semi-circumference to radius = π (where $\pi = 3.1416\cdots = 3\frac{1}{7}$ approximately), we have

π radians = 180°, and hence 1 radian = about 57.3°.

19. The *radian* is especially important in problems concerning the motion of a particle in a circular path. Thus, if

r ft. = radius of the circle,
s ft. = length of arc traversed, and
θ radians = angle swept over by the moving radius, then

$$s = r\theta.$$

This important equation is not true unless the angle is measured in radians. Again, if

v ft. per sec. = linear velocity of the particle in its path, and
ω radians per sec. = its angular velocity, then

$$v = r\omega.$$

Further, if the angular velocity = ω radians per sec. = N rev. per min., then the relation between the numbers ω and N is given by

$$\omega = \frac{\pi N}{30}.$$

In all higher mathematics, when a letter is used for an angle, without designating the unit, it is understood that the letter means the number of radians in the angle.

20. *Definition of sine, cosine and tangent of any angle.*— Let x be any angle, swept over by a moving radius revolving from OX to OL, and suppose for convenience of language that OX extends horizontally to the right. Assume, for the moment that OX and OL are not perpendicular. From any point P of the moving radius drop a perpendicular on the initial line (or the initial line produced), thus forming a right tri-

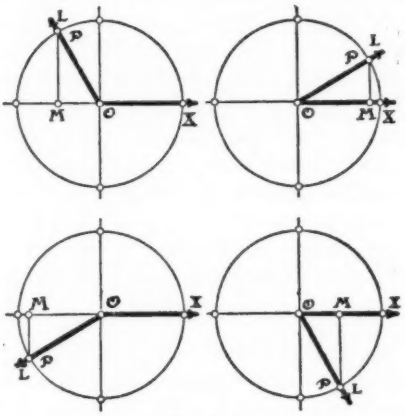


FIG. 16.

angle, called the *triangle of reference* for the given angle x. In this triangle, the perpendicular MP is called the *side opposite O*, and is positive if it runs up, negative if it runs down; the base OM is called the *side adjacent* to O, and is positive if it runs to the right, negative if it runs to the left, and the radius OP is called the *hypotenuse* of the triangle and may always be taken as positive. The *sine*, *cosine* and *tangent* of the angle x are then defined as follows:

$$\sin x = \frac{\text{side opp.}}{\text{hypot.}}, \quad \cos x = \frac{\text{side adj.}}{\text{hypot.}}, \quad \tan x = \frac{\text{side opp.}}{\text{side adj.}} = \frac{\sin x}{\cos x}$$

These ratios are positive or negative numbers, depending only on the position of the terminal side of the angle x, and

are called trigonometric functions of x. The functions of any angle congruent to x are the same as the functions of x, so that we need consider only the angles in "the first revolution," that is, angles between 0° and 360°.

21. To trace the changes in each function as the angle changes from 0° to 360°, draw the figure so that the donominator of the ratio is kept constant, say equal to 1 inch, and trace the changes in the numerator (Fig. 17 for the sine and cosine; Fig. 18 for the tangent). Obviously, the sine will be positive for angles in the upper quadrants; the cosine will be positive for angles in the right hand quadrants; and the tangent will be positive in quadrants *I* and *III*.

The definitions of the functions of 0°, 90°, 180°, and 270°, which were not included above, can now be readily obtained by noting what becomes of the function of a variable angle x when x approaches one of these values as a limit.

In using the "circle of reference" be careful to have every angle start from the initial line that extends horizontally to the right.

Other Trigonometric Functions.

22. *Definition of other trigonometric functions.*—Besides the sine, cosine, and tangent, other functions in common use are the *cotangent*, the *secant*, and the *cosecant*, which are most conveniently defined thus:

$$\cot x = \frac{1}{\tan x}, \quad \sec x = \frac{1}{\cos x}, \quad \csc x = \frac{1}{\sin x}.$$

Less important, but often convenient, are the *versed sine* and the *conversed sine:*

$$\operatorname{vers} x = 1 - \cos x, \quad \operatorname{covers} x = 1 - \sin x.$$

23. It is worth remembering that the sine and cosine are always less than (or equal to) 1, in absolute value; their reciprocals, the secant and cosecant, are always greater than (or equal to) 1, in absolute value; the tangent and cotangent may have any value, positive or negative; while the versed sine and coversed sine are always positive, ranging from 0 to 2.

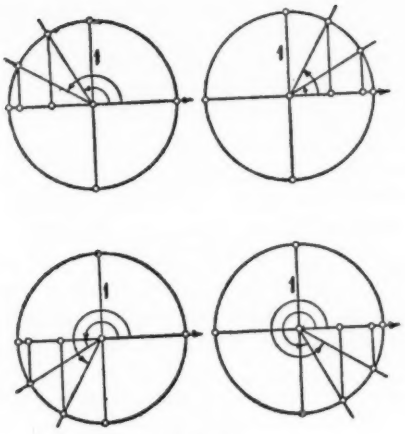


FIG. 17.

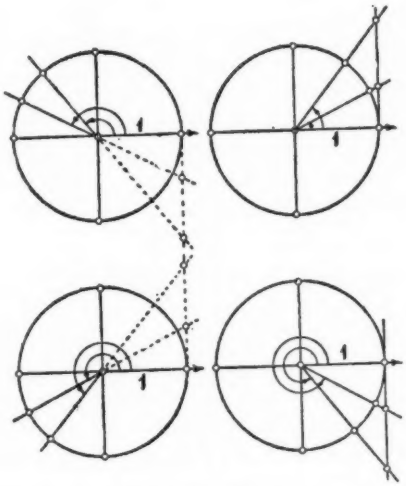


FIG. 18.

24. *Use of the tables: reduction to the first quadrant.*—The tables in common use give the values of the functions only for angles between 0° and 90°, that is, only for angles in the first quadrant. To find the functions of an angle x in one of the other quadrants, find first the "reduced angle" in quadrant *I* (that is, $x - 90°$, or $x - 180°$, or $x - 270°$), and then proceed as in the following examples:*

(*a*) To find $\cos x$, when x is in quadrant *II*. Draw any angle in quadrant *II* to represent the angle x (avoiding, however, lines near the middle of the quadrant) and draw the "reduced angle" $x - 90°$ in quadrant *I*. Then, pointing at the figure, $\cos x$ is *this* line (VVV) [divided by the radius], which is the same in length as *this* line (≋) [divided by the radius], which is the *sine* of $x - 90°$; but the first line is negative; hence

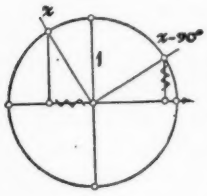


FIG. 19.

$$\cos x = -\sin (x - 90°),$$

where $\sin (x - 90°)$, of course, can be found in the table.

(*b*) To find $\tan x$, when x is in quadrant *II*. Pointing at the figure, $\tan x$ is this line (≋) divided by this line (|||), which is the same as this line (VVV) divided by this line (≡), which is the cotangent of $(x - 90°)$; but the signs are unlike; hence

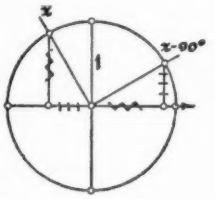


FIG. 20.

$$\tan x = -\cot (x - 90°),$$

where $\cot (x - 90°)$ can be found from the table.

Similarly for any other case.

25. The converse problem of finding the angle corresponding to any given function is complicated by the fact that there will be (in general) *two* angles between 0° and 360° corresponding to any given function. The most satisfactory way

* The given angle is supposed to be already reduced to an angle between 0° and 360°.

to find these two angles, in any numerical case, is to draw the figure, and proceed as in the examples below, in which x_0 in each case represents an angle in the first quadrant which can be found in the table.

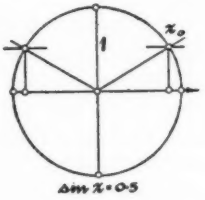


FIG. 21.

Given $\sin x = 0.5$;
$x = x_0$ or $180° - x_0$.

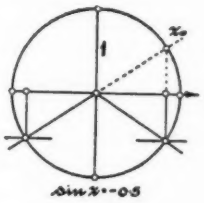


FIG. 22.

Given $\sin x = -0.5$;
$x = 180° + x_0$ or $360° - x_0$.

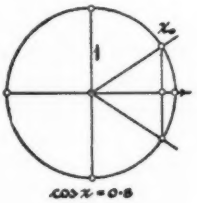


FIG. 23.

Given $\cos x = 0.8$;
$x = x_0$ or $360° - x_0$.

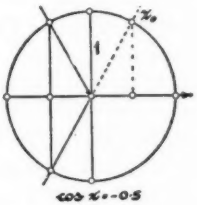


FIG. 24.

Given $\cos x = -0.5$;
$x = 180° - x_0$ or $180° + x_0$.

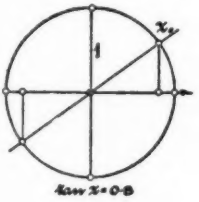


FIG. 25.

Given $\tan x = 0.8$;
$x = x_0$ or $180° + x_0$.

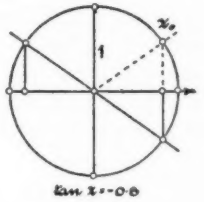


FIG. 26.

Given $\tan x = -0.8$;
$x = 180° - x_0$ or $360° - x_0$.

These results are not formulæ to be memorized; it is much safer, and more intelligent, to draw the appropriate figure, or to visualize it in the mind, for each case as it arises. The student should be thoroughly drilled in numerical cases, especially for angles in the second quadrant.

Notice that an angle is completely determined when we know the value of any one of its functions, and the *sign* of any other function (not the reciprocal of the first).

It we restrict ourselves to angles between 0° and 180°, as in the case of angles in a triangle, then an angle is wholly determined by either its cosine or its tangent; but there will be two angles, x and $180° - x$, corresponding to a given sine.

26. The functions of certain angles in the later quadrants, corresponding to 30°, 45°, and 60° in quadrant *I*, may be found exactly, without the use of the tables, by inspection of the figure (see § 4).

For example, $\cos 120° = -\frac{1}{2}$.

27. If it is required to find the remaining functions of an angle when one function is given, draw a right triangle and proceed as in § 5, considering only the absolute values of the quantities, without regard to sign; then adjust the sign of the answer according to the quadrant in which the angle lies. Or, the angle may be drawn at once in the proper quadrant.

Solution of Oblique Triangles.

28. In any plane triangle the following theorems are easily proved from a figure:

(1) *The "Law of Sines."*—Any side is to any other side as the sine of the angle opposite the first side is to the sine of the angle opposite the other side; in the usual notation:

$$\frac{a}{b} = \frac{\sin A}{\sin B},$$

with two analogous formulæ obtained by "advancing the letters."

(2) *The "Law of Cosines."*—The square of any side is equal to the sum of the squares of the other two sides, minus twice their product times the cosine of the included angle:

$$a^2 = b^2 + c^2 - 2bc \cos A,$$

with two analogous formulæ obtained by "advancing the letters."

These two laws, with the fact that the sum of the angles is 180°, suffice to "solve" any plane triangle, and are important in many theoretical considerations.

The following formulæ which are especially adapted to logarithmic computation, give the tangents of the half-angles in terms of the sides, and are included here for reference:

$$\tan \frac{A}{2} = \frac{r}{s-a}, \qquad \tan \frac{B}{2} = \frac{r}{s-b}, \qquad \tan \frac{C}{2} = \frac{r}{s-c}$$

where

$$s = \frac{a+b+c}{2}$$

and

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} = \text{radius of inscribed circle.}$$

From these formulæ we have at once,

$$\text{Area} = rs = \sqrt{s(s-a)(s-b)(s-c)}.$$

29. The only case which is likely to give any difficulty, is the "ambiguous case" in which the given parts are two sides and the angle opposite one of them. Here we must remember, at a certain point in the work, that when the sine of an angle is given, there will be, in general, two angles corresponding to that sine, one the supplement of the other; so that from that point on, the problem breaks up into two separate problems. But if the sine of an angle is 1, then the only value for the angle is 90°; and if the sine is greater than 1, there is no corresponding angle, and the problem is impossible. It is advisable to construct a fairly accurate figure.

30. *Problems in oblique triangles, triangulation, etc.*

In every case at least a rough sketch should be drawn on which the known parts are clearly marked, and a "blank form" for the computation should be made out for the entire problem, before any of the quantities are looked up in the table.

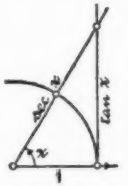
CHAPTER III.

General Properties of the Trigonometric Functions.

31. *Relations between the functions of a single angle.*—The student should convince himself that the following important relations will hold for any angle x:

$$\sin^2 x + \cos^2 x = 1, \quad \tan x = \frac{\sin x}{\cos x},$$

$$\sec^2 x = 1 + \tan^2 x.$$



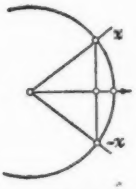
All these relations are easily recalled by the aid of the figures.

Somewhat less important is the following:

$$\csc^2 x = 1 + \cot^2 x.$$

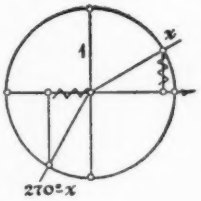
32. *Functions of* $(-x)$. From the figure,

$$\sin(-x) = -\sin x,$$
$$\cos(-x) = \cos x,$$
$$\tan(-x) = -\tan x.$$



33. *Functions of* $(90° + x)$, $(x + 180°)$, *etc.*—Any function of a combination like $(x \pm n90°)$ or $(n90° \pm x)$ can be expressed in terms of a function of x by the use of the figure.

For example, find $\sec(270° - x)$. Take as x any small angle in the first quadrant, and draw the angle $270° - x$. Then, $\sec(270° - x)$ is 1 over the *cosine* of $(270° - x)$, which, pointing at the figure, is the radius over *this* line (VV), which is the same, in length, as the radius over *this* line (VV), which is 1 over the *sine* of x, or $\csc x$. But the signs are opposite; therefore,



$$\sec(270° - x) = -\csc x.$$

This method requires the memorizing of no rules or formulæ, besides the definitions of the functions; a very little practice will develop all the speed and accuracy that can be desired, and the method is one which is readily recalled to mind after long disuse. The special case of complementary angles, however, is worth remembering as a separate formula: Any function of $(90° - x)$ = the co-named function of x.

Formulas for the Sum of Two Angles, Etc.

34. In simplifying trigonometric expressions which occur in calculus, mechanics, etc., the following formulæ are so frequently required that they should be thoroughly memorized. The ability to recognize those relations readily, regardless of the special lettering employed, is a necessary condition for rapid progress in almost any branch of analysis, but *it is highly undesirable to extend the list beyond the limits here given.*

The fundamental formulæ from which all others are derived are these two, the proof of which is obtained from a figure:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y, \tag{1}$$

$$\cos(x + y) = \cos x \cos y - \sin x \sin y. \tag{2}$$

These and the following formulæ should be memorized in words, not in letters: thus, "the sine of the sum of two angles is the sine of the first times the cosine of the second, plus the cosine of the first times the sine of the second," etc.

Dividing (1) by (2) and then dividing numerator and denomerator by the product of the cosines, we have

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}. \tag{3}$$

Changing the sign of y in these three formulæ, and remembering the relations for negative angles, we have the corresponding formulæ for $\sin(x - y)$, $\cos(x - y)$, $\tan(x - y)$, which will be exactly the same as (1), (2), and (3) with all the connecting signs reversed:

$$\sin(x-y) = \sin x \cos y - \cos x \sin y, \tag{4}$$

$$\cos(x-y) = \cos x \cos y + \sin x \sin y, \tag{5}$$

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}. \tag{6}$$

Putting $x = y$ in (1), (2), and (3) we have at once

$$\sin 2x = 2 \sin x \cos x, \tag{7}$$

$$\begin{aligned} \cos 2x &= \cos^2 x - \sin^2 x \\ &= 1 - 2\sin^2 x = 2\cos^2 x - 1, \end{aligned} \tag{8}$$

$$\tan 2x = \frac{2 \tan x}{1 - \tan^2 x}. \tag{9}$$

Solving (8) first for $\sin x$ and then for $\cos x$, and putting $2x = y$, or $x = y/2$, we find

$$\sin \frac{y}{2} = \pm \sqrt{\frac{1 - \cos y}{2}}, * \tag{10}$$

$$\cos \frac{y}{2} = \pm \sqrt{\frac{1 + \cos y}{2}}, \tag{11}$$

whence,

$$\tan \frac{y}{2} = \pm \sqrt{\frac{1 - \cos y}{1 + \cos y}}. \tag{12}$$

This last formula may be transformed, by rationalizing numerator or denominator, into

$$\tan \frac{y}{2} = \frac{1 - \cos y}{\sin y} = \frac{\sin y}{1 + \cos y}.$$

Other formulas, useful for special purposes, should not be memorized, but should be derived as needed.

35. In proving the identity of two trigonometric expressions, it is best to reduce each expression separately to its simplest form.

* The *plus* sign is to be used when $\sin \frac{1}{2}y$ is positive, the *minus* sign when $\sin \frac{1}{2}y$ is negative. Similarly in the next two formulas.

The fallacy of supposing that because a true relation can be deduced from a given equation, the given equation is therefore necessarily true, should be carefully explained.

For example, from the false equation $3 = -3$ we can obtain the true equation $9 = 9$ by squaring both sides; or, from the false equation $30° = 150°$ we can obtain the true equation $\frac{1}{2} = \frac{1}{2}$ by taking the sine of both sides; but in each of these cases the step taken is not reversible.

36. The following device for transforming an expression of the form $a \cos x + b \sin x$ is often useful:

$$a \cos x + b \sin x = \sqrt{a^2 + b^2}\left[\frac{a}{\sqrt{(a^2 + b^2)}} \cos x + \frac{b}{\sqrt{(a^2 + b^2)}} \sin x\right]$$

$$= A \cos (x - B),$$

where $$A = \sqrt{(a^2 + b^2)} \text{ and } \tan B = \frac{b}{a}.$$

37. *The inverse functions.*

The angle between $-90°$ and $+90°$ whose sine is x is denoted by $\sin^{-1} x$.*

The angle between $0°$ and $180°$ whose cosine is x is denoted by $\cos^{-1} x$.

The angle between $-90°$ and $+90°$ whose tangent is x is denoted by $\tan^{-1} x$.

In simplifying expression involving these "inverse functions," it is well to take a single letter to stand for each inverse function; as, $y = \sin^{-1} x$, whence, by definition, $\sin y = x$; etc.

38. *Solution of trigonometric equations.* Many trigonometric equations can be solved only by the "method of trial and error" (see chapter on numerical computation). In other cases, however, it is possible, by the use of the formulas given above, to transform the given equation into a form involving only a single function of a single angle; if this equation can be solved for the function in question, then the required value (or values) of the angle can be found from the tables or it can be shown that no solution exists.

* The symbol $\sin^{-1} x$ (or arc sin x) is often defined as simply "the angle whose sine is x"; but since there are many such angles, it is necessary to specify which one is to be taken as "the" angle, if the symbol is to have any definite meaning.